CLASSIFICATION OF THE UNITARY IRREDUCIBLE REPRESENTATIONS OF THE SU(1,1) GROUP

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Chapter 1

Introduction

The unitary irreducible representations (UIR) of the Lorentz group in two space and one time , were investigated by V. Bargmann[1] and I. M. Gelfand[2] independently of each other. The subject matter of this review work is based on the results of these investigations. The Lorentz group in two space and one time consists of all linear transformations that leaves the quadratic form $x_1^2 + x_2^2 + x_3^2$ invariant.Since we shall be interested in the half integral representations also we shall work with the spinor group of the Lorentz group. The spinor group of the Lorentz group is the $SU(1,1)$ group and consists of all pseudo unitary matrices of determinant $+1$. The $SU(1,1)$ group being a noncompact group , all its UIR's are infinite dimensional except for the trivial one dimensional case. Though the purpose of this work is to study the $SU(1,1)$ group we work on the $SU(2)$ group also side by side. The $SU(2)$ group which is the spinor group for space rotations is a well studied group. Thus working on the $SU(2)$ group gives us a warm-up for dealing with the $SU(1,1)$ group.

Our plan of investigation shall be to firstly specify the group action. This amounts to specifying the carrier space also. Secondly from the group action we determine the infinitismal generators of the group, which in turn gives the commutation relations. Finally using Schur's lemma we venture into the classification and realization of the UIR's of the group.

For the purpose of the classification of the UIR's of the $SU(1,1)$ group we realize the group as transformations on the Hilbert space. The specification of the group action for the $SU(1,1)$ group amounts to specifying the group action, the function

space and the scalar product in the function space. During classification we donot specify these explicitly.The $SU(1,1)$ group sports three infinitismal generators x_1, x_2 and x_3. Using Schur's lemma the invariant operator $Q = x_1^2 + x_2^2 + x_3^2$ turns out to be a scalar. This quantity is used as a label for the UIR's . In $SU(1,1)$ unlike $SU(2)$ the invariant operator itself is not enough to specify the UIR's . It is seen that in $SU(1,1)$ we need another parameter λ_0 to label the UIR's. We shall confine ourselves to $\lambda_0 = \quad 0, \quad 1/2$ which amounts to confining to integral and half integral representations. The UIR's are classified based on the set of values that q,λ_0 and m can take. The UIR's are broadly classified into continuous class and discrete class. In the continous class q takes a continuous set of values in a domain while in the discrete class q is restricted to take discrete values in a domain. The exact classification is (i) Continous Class: (a) Principal series : $\lambda_0 = 0, 1/2 \quad ; q \geq 1/4 \quad ; m = 0, \pm 1/2, \pm 1, ...$ (b) Supplementary series : $\lambda_0 = 0 \quad ; 0 \leq q \leq 1/4 \quad ; m = 0, \pm 1, \pm 2, ...$ (ii) Discrete class : (a) trivial one dimensional case: $\lambda_0 = 0, 1/2 \quad ; q = 0 \quad ; m = 0$ (b)negative discrete series: $\lambda_0 = 0, 1/2 \quad ; k = 1/2, 1, 3/2, ... \quad ; m = -k, -k-1, -k-2, -k-3...$ (c) positive discrete series: $\lambda_0 = 0, 1/2 \quad ; k = 1/2, 1, 3/2, ... \quad ; m = k, k+1, k+2, ...$

Chapter 2

Introduction to Group Theory

2.1 Introduction

This chapter deals with the specification of the group which will be dealt by us.A group is specified by defining the binary funtion.In the following section, we define a group. In the next section we show that every one parameter group is equivalent to a group of translation.It will be seen that this means to saying that we can always start with a set of canonical parameters.In section 2.4 we clearly specify the group which will be dealt, by demanding a specific form of the generators.The form of the binary funtion which specifies our group is determined.Finally we generalise the whole discussion for a n parameter group.

2.2 Group

Consider a set G,whose elements are denoted by a,b,... .Let each group element be specified by n real variables (called the group parameters).The parameters corresponding to an element a shall be denoted by a^i(i=1 to n) .Thus we can write

$$a = a(a^i) \qquad ,\forall, \qquad a \in G \tag{2.1}$$

The domain of the parameters decides whether the group is compact,noncompact, continuous or noncontinuous.The space spanned by a^i is also called the group manifold.

Let us demand that this set G forms a group under an appropriate binary operator

(binary operator is an operator defined as $f : GXG \rightarrow G$).This amounts to demanding that the elements of the group satisfy the four axioms of the group given below.

(i) For any two elements a,b we define the analytic binary function

$$\phi(a, b) \tag{2.2}$$

such that

$$c = \phi(a, b) \in G \tag{2.3}$$

(ii) For any three elements a,b,c the binary function defined is associative.(i.e) if a,b,$c \in G$ then

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c) \tag{2.4}$$

(iii) For any element a, there exists a unique element (the identity element), denoted by e, such that

$$\phi(a, e) = \phi(e, a) \tag{2.5}$$

(iv) For any element a, we have a corresponding element a^{-1} (the inverse of a), such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = e \tag{2.6}$$

2.3 One Parameter Group and Translation

In this subsection we shall show that every one parameter group is equivalent to a group of traslations.For this purpose, we shall firstly find the partial derivative of the binary function with respect to the group parameter.From the expression for the derivative, a few steps show that it is equivalent to a translation.

Consider the one parameter binary function

$$c = \phi(b, a) \tag{2.7}$$

This can be visualised as ,the element a acting on the element b to give the new element c . If we make a small variation δa in the element a , it will give a new

element $c + \delta c$. In the figure it can be easily observed that, there exists an element $\bar{\delta}a$ which operates on c to give $c + \delta c$.

Thus we have the following relations,

$$c = \phi(b, a) \tag{2.8}$$

$$c + \delta c = \phi(c, \bar{\delta}a) \tag{2.9}$$

$$c + \delta c = \phi(b, a + \delta a) \tag{2.10}$$

Expanding eqn.[2.9],gives

$$\delta c = \frac{\partial \phi(c, a)}{\partial a} \mid_{a=e} .\bar{\delta}a \tag{2.11}$$

Now starting from eqn.[2.9] ,and using eqn.[2.8] in it we have

$$c + \delta c = \phi(\phi(b, a), \bar{\delta}a) \tag{2.12}$$

Since the binary function is associative, we have

$$c + \delta c = \phi(b, \phi(a, \bar{\delta}a) \tag{2.13}$$

Comparing eqn.[2.13] and eqn.[2.10] gives

$$a + \delta a = \phi(a, \bar{\delta}a) \tag{2.14}$$

Expanding the above eqn. we have

$$\delta a = \frac{\partial \phi(a, b)}{\partial b} \mid_{b=e} .\bar{\delta}a \tag{2.15}$$

or,

$$\bar{\delta}a = (\frac{\partial \phi(a, b)}{\partial b} \mid_{b=e})^{-1} .\delta a \tag{2.16}$$

Using eqn [2.16] in eqn[2.11],we have

$$\delta c = (\frac{\partial \phi(c,a)}{\partial a} \mid_{a=e}).(\frac{\partial \phi(a,b)}{\partial b} \mid_{b=e})^{-1}.\delta a \tag{2.17}$$

Denoting

$$u(c) = \frac{\partial \phi(c,a)}{\partial a}|_{a=e} \tag{2.18}$$

$$\psi(a) = (\frac{\partial \phi(a,b)}{\partial b}|_{a=e})^{-1} \tag{2.19}$$

Thus eqn[2.17] has the compact form

$$\delta c = u(c).\psi(a).\delta a \tag{2.20}$$

This is the relation which gives the partial derivative of the binary function with respect to the group parameter. Starting from eqn[2.20] we shall in a few steps show that any one parameter group is equivalent to the group of translations.It can be easily verified that

$$\psi(a) = [u(a)]^{-1} \tag{2.21}$$

or,

$$\psi(a).u(a) = 1 \tag{2.22}$$

Using eqn [2.22] in eqn[2.20], we have

$$u(a).\delta c = u(c).\delta a \tag{2.23}$$

(i.e)

$$[u(c)]^{-1}.\delta c = [u(a)]^{-1}.\delta a \tag{2.24}$$

Integrating both sides we have

$$\int_b^c [u(c)]^{-1}.\delta c = \int_0^a [u(a)]^{-1}.\delta a \tag{2.25}$$

now denoting

$$t(a) = \int [u(a)]^{-1}.\delta a \tag{2.26}$$

and normalising such that

$$t(e) = 0 \tag{2.27}$$

and using eqn [2.26] we have

$$t(c) = t(b) + t(a) \tag{2.28}$$

Thus we have shown that any one parameter group is equivalent to a group of translation. Conversely we can state that, for a one parameter group defined by the binary function

$$c = \phi(a, b) \tag{2.29}$$

there always exists a canonical parameter t related to the initial parameter by

$$a = a(t) \tag{2.30}$$

(in arriving at the above relation we have required that eqn[2.26] has an inverse relation) such that,

$$a(t_1 + t_2) = \phi(a(t_1), a(t_2)) \tag{2.31}$$

where

$$c = a(t_1 + t_2) \tag{2.32}$$

$$a = a(t_1) \tag{2.33}$$

$$b = a(t_2) \tag{2.34}$$

2.4 Generators of a One Paramater Group

In the last section we have seen that the parameter a of a one parameter group can be given in terms of the canonical parameter t (eqn.[2.30]) by

$$a = a(t) \tag{2.35}$$

This relation between the canonical parameter t and the original parameter a is defined as the generators of the one parameter group. It is similar to the concept of generating functions of the Bessel functions in the study of special functions.

In general the generators $a(t)$ given by eqn.[2.35] can have any functional form depending on defnition of our binary function in eqn.[2.3] . But we shall confine ourselves to only those binary functions which give the generators of the form

$$a = a(t) = \exp(At) \tag{2.36}$$

where A is independent of t.This obviously restricts ourselves to a special form of the binary function.This special form of the of the binary function turns out to be

$$a(t+t') = \phi(a(t), a(t')) \tag{2.37}$$

$$= a(t).a(t') \tag{2.38}$$

Let us explicitly show that the choice of binary function of the form eqn.[] leads to the generator of the group of the form eqn.[]. Starting from eqn[] we can write

$$a(t+\delta t) = a(t).a(\delta t) \tag{2.39}$$

Expanding $a(t+\delta t)$ and $a(\delta t)$, we have

$$a(t) + \frac{\partial a(t)}{\partial t}.\delta t = a(t)[a(0) + \frac{\partial a(t)}{\partial t} \mid_{t=0} .\delta t] \tag{2.40}$$

Thus we have

$$\frac{\partial a(t)}{\partial t} = a(t).\frac{\partial a(t)}{\partial t}|_{t=0} \tag{2.41}$$

Integrating eqn.[2.41] we get

$$a(t) = \exp(t.\frac{\partial a(t)}{\partial t}|_{t=0}) \tag{2.42}$$

$$= \exp(At) \tag{2.43}$$

where

$$A = \frac{\partial a(t)}{\partial t}|_{t=0} \tag{2.44}$$

is called the infinitismal generator of the group.Thus eqn[2.36] is the solution of eqn.[2.38].

To summarise, we confine ourselves to groups, for which the binary function is defined by eqn.[2.38].As a cosequence, the generator of the group always has the form given by eqn[2.36].

2.5 n-Parameter Group

We now generalise all the above discussions for a one parameter group to n-parameter group. Let the group element a, be related to the n real parameters a^p by

$$a = a'(a^p) \tag{2.45}$$

Thus the binary function corresponding to eqn[2.7] takes the form

$$c = a'(c^i) = \phi'(a'(a^p), a'(b^q)) \tag{2.46}$$

which in component form is

$$c^i = \phi^i(a^p, b^q) \tag{2.47}$$

All the indices i,p,q run from 1 to n.The corresponding relation which gives the partial derivative with respect to the group parameters is

$$\delta c^i = u^i_s(c)\psi^s_k(a)\delta a^k \tag{2.48}$$

where

$$u^i_s(c) = \frac{\partial \phi^i(a^p, c^q)}{\partial a_s} \mid_{a^p=e} \tag{2.49}$$

$$\psi^s_k(a) = [u^s_k(a)]^{-1} \tag{2.50}$$

Starting from eqn[2.48] and proceeding as we did in the one parameter case, we get the canonical parameters as

$$t^r = t^r(a^p) \tag{2.51}$$

where each component of t obeys translation rules.The inverse of eqn[2.51] gives

$$a^p = a^p(t^r) \tag{2.52}$$

Thus for any group element a, we have

$$a = a'(a^p) \tag{2.53}$$

$$= a'(a^p(t^r)) \tag{2.54}$$

$$\equiv a(t^r) \tag{2.55}$$

The final relation given by eqn[2.55] is the relation between an element a of the n-parameter group and the n canonical parameters t^r. Thus eqn[2.55] is nothing but the relation giving the generators of the group.

In the one parameter case we restricted ourselves to generators of the form given by eqn[2.36] and we got the corresponding restriction on the binary function given by eqn [2.38].For n-parameter group we restrict ourselves to the generators of the form

$$a = a(t_r) = \prod_{i=1}^{n} \exp(A_i t_i) \tag{2.56}$$

The binary function which gives eqn[2.56] as the solution is

$$a(t_r + t_r{}') = \phi(a(t_r), a(t_r{}')) \tag{2.57}$$

$$= a(t_r).a(t_r{}') \tag{2.58}$$

To summarise , we confine ourselves to n-parameter groups, for which the binary function is defined by eqn[2.57].As a consequence ,the generator of the group has the form given by eqn[2.56].

The demand made in eqn[2.57] is equivalent to requiring the n-parameter group to be expressible as the direct product of n single parameter subgroups.

Later in Chapter 3, when we look for the realization of the group as transformations on the coordinate space given by

$$\bar{x} = T(a, x) \tag{2.59}$$

The above restriction allows us to have only linear transformations, (i.e)

$$T(a, x) = U(a).x \tag{2.60}$$

Similarly in Chapter 4, when we realize the group as transformations

on the function space given by

$$f'(x) = T(a, f(x)) \tag{2.61}$$

The above restriction allows us to have only two forms of T namely

$$(i) T(a, f(x)) = f(a^T.x) \tag{2.62}$$

$$(ii) T(a, f(x)) = \mu(a, a^T.x).f(a^T.x) \tag{2.63}$$

2.6 Some Results

By now we have made it clear that the group action that will be considered is

$$\phi(a(t_r), a(t_r{}') = a(t_r).a(t_r{}') \tag{2.64}$$

The generator of the group is

$$a(t_r) = \prod_{i=1}^{n} \alpha_i \tag{2.65}$$

We now list a few theorems without their proofs.

Theorem 1 :

The infinitismal generators X^r defined above forms the basis of a n-dimensional vector space (called the Lie Algebra of the group) over the field t_r.

Corollary 1

Every linear combination of X^r with coeficients from the field gives a member of the vector space itself.

Theorem 2 :

The generators obey the commutation rules,

$$[X^i, X^j] = c_k^{ij}.X^k \tag{2.66}$$

where c_k^{ij} , called the structure constants , are independent of the group parameters t^r.

In the study of representations,it will be seen that the commutation rules play a very central role.Given a group our procedure will be to firstly get the commutation relations. Then we make a suitable choice of the Hilbert space in which the determination of the representations is simplified. Thus to summarise,

(i) Given the group action,

$$\phi(a(t_r), a(t_r{}')) = a(t_r).a(t_r{}') \tag{2.67}$$

find the one parameter subgroups whose direct product is equivalent to the above group action.

(ii) Determine the generators X^r

(iii) Get the commutation relations

$$[X^i, X^j] = c_k^{ij}.X^k \tag{2.68}$$

In the next chapter we shall demonstrate the first three steps for the groups that will be dealt by us, namely , the rotation group in 3-dimensions, Lorentz group in 2-space and 1-time and their corresponding spinor groups.The representation theory will be dealt in later Chapters.

Chapter 3

The SU(1,1) Group

3.1 Introduction

A general line of approach for any group was given in the first Chapter. In this Chapter we shall use this approach in the study of SO(3) ,SU(2), SO(2,1) and SU(1,1).Each of these groups are realized as transformations on the coordinate space.In the following Chapter we shall realize these groups as transformations on the function space.By realization we mean a group which is isomorphic to the original group.

3.2 Realization of the Group as Transformations on the Coordinate Space

Consider the transformation,

$$\bar{x}^i = T^i(a_r, x_j) \tag{3.1}$$

where x_j are the components of a vector in the n-dimensional coordinate space and a_r are the group parameters.We assume that the function T^i are analytic.If the above realization as transformations is to form a group, we require,

$$\bar{x} = T(a, x) \tag{3.2}$$

$$\bar{\bar{x}} = T(b, \bar{x}) \tag{3.3}$$

$$\bar{\bar{x}} = T(c, x) \tag{3.4}$$

therefore

$$T(c, x) = T(b, T(a, x)) \tag{3.5}$$

$$= T(\phi(a,b),x) \tag{3.6}$$

(i.e)

$$T(b,T(a,x)) = T(\phi(a,b),x) \tag{3.7}$$

The eqn [3.7] completely defines the group action.It should be noted that $\phi(a,b)$ depends only on the group parameters

In Chapter 1 we had mentioned that we will be confining ourselves to those group actions for which the generators take the exponential form.This restriction in the above realization confines ourselves to deal with linear transformations only.

Thus we shall be dealing with the transformation

$$\bar{x}^i = a^i_j(t_r).x^j \tag{3.8}$$

where t_r are the canonical group parameters.The generators for the above group action is given by

$$X^k = \frac{\partial a^i_j(t_r)}{\partial t^k} \mid_{t_r=e} \tag{3.9}$$

The linear transformation given by eqn [3.8] is nothing but a matrix a^i_j multiplying a column matrix x^i to give the transformed column matrix $\bar{x}^i$. The elements of the matrix a^i_j are dependent on the canonical group parameters t_r .We shall now mention a bit about some nomenclature which is based on the properties of the matrix a^i_j.The most general group is the $GL(n,C)$.Here G stands for 'general',L reminds that the transformation is linear,C states that the matrix elements are complex and n stands for the dimension of the matrix. Nextly we have $SL(n,C)$ where S implies that every matrix is unimodular.We also have $SO(n)$ and $SU(n)$ where O denotes that the matrices are orthogonal and U denotes that the matrices are unitary.Thus we have $GL(n,C)$, $GL(n,R)$ $SL(n,C), SL(n,R), O(n), SO(n), U(n), SU(n)$. In our study we shall be dealing with $SO(3), SU(2), SO(2,1)$ and $SU(1,1)$.

3.3 The Rotation Group - SO(3)

The $SO(3)$ group consists of all linear transformation that leaves the quadratic form

$$\delta_{ij}.x^i.x^j \tag{3.10}$$

invariant.Equivalently it consists of all 3×3 orthogonal matrices of determinant +1. The orthogonality relation is

$$a^i_j.b^j_k = \delta^i_k \tag{3.11}$$

Every rotation in the three dimentional coordinate space corresponds to an element in $SO(3)$.Also the elements of $SO(3)$ can be mapped on to the points on a sphere.Let us start with a vector of modulus r in the coordinate space.If we apply every element of $SO(3)$ to this vector,all the transformed vectors again have modulus r due to the invarience of the form.The locus of the tip of the transformed vectors lies on a sphere.

(i)The group action is defined as

$$\bar{x}^i = a^i_j.x^j \tag{3.12}$$

$$\bar{\bar{x}}^i = b^i_j.\bar{x}^j \tag{3.13}$$

$$\bar{\bar{x}}^i = c^i_j.x^j \tag{3.14}$$

from these equations we have

$$c^i_k = a^i_j.b^j_k \tag{3.15}$$

where a, b, c are orthogonal matrices

(ii) Our next aim is to find the one parameter subgroups of the rotation group.It should be mentioned here that depending on the choice of the canonical parameters we can have various parametrizations of the group. We shall be considering the θ,ϕ,ψ parametrization. Another very popular parametrization is the Euler parametrization . In the θ,ϕ ,ψ parametrization every element of $SO(3)$ can be written in terms of the one parameter subgroups.(i.e)

$$a^i_j(\psi,\phi,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix} . \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.16}$$

It should be noted that for the ϕ parameter the inverse subgroup has been used.This is a general convention used to keep the commutation relations of $SO(3)$ right handed. Later when we determine the $SU(2)$ subgroups corresponding to the $SO(3)$ subgroups the ϕ parameter subgroup used is the inverse of the one given in eqn.[3.16].

(iii) Once the parametrization is done, we should find the infinitismal generators.

$$X_1 = \frac{\partial a^i_j}{\partial \psi} \mid_{\psi=\phi=\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{3.17}$$

$$X_2 = \frac{\partial a^i_j}{\partial \phi} \mid_{\psi=\phi=\theta=0} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.18}$$

$$X_3 = \frac{\partial a^i_j}{\partial \theta} \mid_{\psi=\phi=\theta=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.19}$$

Using the above relation, we can write the generator of the group elements as

$$a^i_j(\psi\phi\theta) = \exp(X_1\psi).\exp(X_2\phi).\exp(X_3\theta) \tag{3.20}$$

(iv) After the determination of the infinitismal generators we find the commutation relations by blunt calculation.The infinitismal generators are found to obey the following commutation relations.

$$[X^i, X^j] = -\epsilon^{ij}_k . X^k \tag{3.21}$$

where ϵ^{ij}_k is the antisymmetric tensor.

3.4 The Spinor Group for Rotations - SU(2)

The $SU(2)$ group consists of all uniatry matrices of determinant $+1$. A general unitary matrix has the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{3.22}$$

with $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ where α and β are complex numbers. The rotation group $SO(3)$ is doubly covered by the $SU(2)$ group. The purpose of dealing with the $SU(2)$ group is, as we shall see in the Chapter on representations,that it gives the half integral representations also.

(i) Our first aim should be to define the group action.Presently for $SU(2)$ we shall not use the carrier space to be the complex space in 2-dimentions. Our carrier space shall be the set of 2×2 traceless hermitian matrices of the form

$$\begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} \tag{3.23}$$

This deviation from the standard approach,as we shall be soon see,will lead to an easier way to map $SO(3)$ onto $SU(2)$.Denoting the traceless hermitian matrix by X, and the group elements by u, we define the group action as,

$$X' = u^\dagger X u \tag{3.24}$$

It can be easily verified that X' is also a traceless hermitian matrix.

$$(X')^\dagger = (u^\dagger X u)^\dagger = u^\dagger X^\dagger u = u^\dagger X u = X' \tag{3.25}$$

$$Tr(X') = Tr(u^\dagger X u) = Tr(X u^\dagger u) = Tr(X) = 0 \tag{3.26}$$

Thus X' is also a member of the carrier space. Similarly we can show that the defined group action obeys the axioms of the group.

$$X' = u_a^\dagger X u_a \tag{3.27}$$

$$X'' = u_b^\dagger X' u_b \tag{3.28}$$

$$X'' = u_c^\dagger X u_c \tag{3.29}$$

Therefore

$$X'' = u_b^\dagger u_a^\dagger X u_a^\dagger u_b^\dagger = (u_a u_b)^\dagger X (u_a u_b) \tag{3.30}$$

(i.e)

$$u_c = u_a u_b \tag{3.31}$$

A good discussion of these points is given in the Chapter 4 of Goldstein[3].

(ii) Once the group action is defined our next aim should be to express the group elements in terms of the one parameter subgroups.For this we need to determine the one parameters subgroups, which unlike $SO(3)$ is not at all obvious.The one parameter

subgroups are determined by seeing the subgroups in $SU(2)$ corresponding to the one parameter subgroups in $SO(3)$.

Consider the group actions in $SO(3)$ and $SU(2)$

$$x' = a.x \tag{3.32}$$

$$X' = u^{\dagger} X u \tag{3.33}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.34}$$

and

$$X = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} \tag{3.35}$$

Thus it is clear that every element in the carrier space of the $SO(3)$ group corresponds to an element in the carrier space of the $SU(2)$ group. Thus the problem turns down to the determination of the appropriate u in eqn[3.33] which gives the same transformation of x_1,x_2,x_3 as when , the one parameter subgroups are substituted in eqn[3.30]. Consider the subgroup

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.36}$$

Thus we have to find the appropriate u, which satisfies

$$\begin{pmatrix} x_3 & (\cos\theta x_1 + \sin\theta x_2) + i(\cos\theta x_2 - \sin\theta x_1) \\ (\cos\theta x_1 + \sin\theta x_2) - i(\cos\theta x_2 - \sin\theta x_1) & -x_3 \end{pmatrix} = u^{\dagger}. \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} .u \tag{3.37}$$

The appropriate u for the subgroup given in eqn[3.36] turns out to be

$$u = \begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix} \tag{3.38}$$

Similarly calculating the other subgroups we have

$$u(\psi,\phi,\theta) = \begin{pmatrix} \cos\psi/2 & i\sin\psi/2 \\ i\sin\psi/2 & \cos\psi/2 \end{pmatrix} . \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix} . \begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix} \tag{3.39}$$

Here the ϕ parameter subgroup of the $SO(3)$ group used during mapping is the inverse of the one used in eqn[3.16].This is done to get the commutation relations of the $SU(2)$ group in a specified form.In $SO(3)$ the inverse was used to keep the commutation relations of the $SO(3)$ group right handed. At this stage it is worth remarking that if we defined the group action to be

$$X' = uXu^{\dagger} \tag{3.40}$$

instead of eqn[3.24] , then the whole analysis remains the same, if every element of the group is replaced by its inverse.

(iii) We now pursue the determination of the infinitsmal generators.

$$A_1 = \frac{\partial u}{\partial \psi} \mid_{\psi=\phi=\theta=0} = \frac{i}{2}. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.41}$$

$$A_2 = \frac{\partial u}{\partial \phi} \mid_{\psi=\phi=\theta=0} = \frac{i}{2}. \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.42}$$

$$A_3 = \frac{\partial u}{\partial \theta} \mid_{\psi=\phi=\theta=0} = \frac{i}{2}. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.43}$$

Using the sigma matrices, we have

$$A_i = \frac{i}{2}.\sigma_i \tag{3.44}$$

Now using the fact that the sigma matrices are hermitian, we conclude that A_i are skew hermitian matrices.From the well known result that every unitary matrix can be expressed in terms of a exponential of a hermitian matrix given by

$$u = \exp(iH) \tag{3.45}$$

and demanding our generators to be of the form eqn[3.45],we replace A_i byJ_i.Thus we get

$$J_i = -i.A_i \tag{3.46}$$

Thus the generators of the $SU(2)$ group are

$$J_i = \frac{1}{2}.\sigma_i \tag{3.47}$$

In terms of the infinitismal generators, the generator of the group elements can be written as

$$u(\psi, \phi, \theta) = \exp(iJ_1\psi).\exp(iJ_2\phi).\exp(iJ_3\theta) \tag{3.48}$$

(iv) The commutation relations for the infinitismal generators are easily got using the commutation relations for the sigma matrices.(i.e) using

$$[\sigma^i, \sigma^j] = 2i.\epsilon_k^{ij}.\sigma^k \tag{3.49}$$

we have

$$[J^i, J^j] = i.\epsilon_k^{ij}.J^k \tag{3.50}$$

3.5 The Lorentz Group - SO(2,1)

The $SO(2,1)$ group consists of all linear transformations that leaves the quadratic form

$$g_{ij}x^ix^j \tag{3.51}$$

invariant, where,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.52}$$

Equivalently it consists of all 3×3 pseudo orthogonal matrices of determinant $+1$. The orthogonality relation is,

$$g_{ij}.a_m^i.b_n^j = g_{mn} \tag{3.53}$$

We had mentioned that the rotation group can be mapped on to the points on a sphere. Analogously the elements of $SO(2,1)$ can be mapped on to either the points on a hyperboloid, on a paraboloid or on a cone (light cone). Let us start with a space-like vector ($g_{ij}x^ix^j > 0$) in the coordinate space of x_1,x_2,x_3. If we apply every element of $SO(2,1)$ to this vector, all the transformed vectors are again space like due to the invariance of the form. The locus of the tip of the transformed vectors lies on a hyperboloid. Similarly if we start with a time-like vector ($g_{ij}x^ix^j < 0$) the points lie on a paraboloid, and if we start with a light-like vector ($g_{ij}x^ix^j = 0$) the points lie on the light cone. Presently since our carrier space is the coordinate space, this

doesn't lead to complications. Later when we realize the group on the Hilbert space, it requires us to integrate over the whole space and thus all time like , space like and light like vectors are to be taken into consideration at the same time. Presently we need to deal with only one type at a time.

(i) We define the group action to be

$$x'^i = a^i_j.x^j \tag{3.54}$$

Let us assume that this forms a group. For this the defined operation should satisfy the axioms of sec.1.1. Firstly let us show that the group action is a binary function. For this we need to show that

$$c^i_j = a^i_k.b^k_j \tag{3.55}$$

leaves the form given in eqn[3.51] invariant. Consider

$$d = g_{ij}x'^i x'^j \tag{3.56}$$

$$= g_{ij}c^i_m x^m c^j_n x^n \tag{3.57}$$

$$= g_{ij}c^i_m c^j_n x^m x^n \tag{3.58}$$

$$= g_{ij}a^i_p b^p_m a^j_q b^q_n x^m x^n \tag{3.59}$$

$$= g_{ij}a^i_p a^j_q b^p_m b^q_n x^m x^n \tag{3.60}$$

$$= g_{pq}b^p_m b^q_n x^m x^n \tag{3.61}$$

$$= g_{mn}x^m x^n \tag{3.62}$$

Thus the group action satisfies the closure property. Similarly we can show that the identity element is δ^i_j, (i.e) it satisfies

$$a^i_j \delta^j_k = a^i_k \tag{3.63}$$

Similarly the inverse of a is

$$a^{-1} = g.a^T.g \tag{3.64}$$

(i.e)

$$(a^{-1})^i_j = g^{in}.(a^T)_n{}^m.g_{mj} \tag{3.65}$$

$$= g^{in}.a^{m}{}_{n}.g_{mj} \tag{3.66}$$

Now it can be easily shown that a acted on a^{-1} gives the identity.

$$a.a^{-1} = a^{i}_{k}.(a^{-1})^{k}_{j} \tag{3.67}$$

$$= a^{i}_{k}.g_{mj}.g^{kn}.a^{m}_{n} \tag{3.68}$$

$$= g^{kn}.a^{i}_{k}.a^{m}_{n}.g_{mj} \tag{3.69}$$

$$= g^{im}.g_{mj} \tag{3.70}$$

$$= \delta^{i}_{j} \tag{3.71}$$

Thus the group action in eqn[3.54] satisfies the axioms of a group.

(ii)The group elements written in terms of the one parameter subgroups takes the form,

$$a(\psi,\phi,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi \\ 0 & \sinh\psi & \cosh\psi \end{pmatrix}.\begin{pmatrix} \cosh\phi & 0 & -\sinh\phi \\ 0 & 1 & 0 \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix}.\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.72}$$

In the above parameterization going by the convention used in $SO(3)$ we have used the inverse of the ϕ parameter subgroup. But we cannot conclude that in $SO(2,1)$ this has been done for right handedness because now we have two space and one time axis. When we are mapping the subgroups of $SO(2,1)$ onto $SU(1,1)$, unlike $SU(2)$ we shall use the subgroups listed in eqn[3.72] without any change. Also it is clearly observed that $SO(2)$ is a subgroup of $SO(2,1)$.

(iii) The infinitismal generators of the group are,

$$X_1 = \frac{\partial a}{\partial\psi}\mid_{\psi=\phi=\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.73}$$

$$X_2 = \frac{\partial a}{\partial\phi}\mid_{\psi=\phi=\theta=0} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.74}$$

$$X_3 = \frac{\partial a}{\partial\theta}\mid_{\psi=\phi=\theta=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.75}$$

Thus the generator of the group is,

$$a(\psi, \phi, \theta) = \exp(X_1\psi).\exp(X_2\phi).\exp(X_3\theta) \tag{3.76}$$

(iv) The commutation relations satisfied by the infinitismal generators, which are determined by blunt calculation are

$$[X_1, X_2] = X_3 \tag{3.77}$$

$$[X_2, X_3] = -X_1 \tag{3.78}$$

$$[X_3, X_1] = -X_2 \tag{3.79}$$

3.6 The Spinor Group for Lorentz Rotations - SU(1,1)

The $SU(1,1)$ group consists of all pseudo unitary matrices of determinant $+1$. The pseudo unitarity is given by

$$u\sigma_3 u^\dagger = \sigma_3 \tag{3.80}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.81}$$

An element of $SU(1,1)$ has the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{3.82}$$

with $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$ where α and β are complex numbers. $SU(1,1)$ is the covering group of $SO(2,1)$ and leads to representations that are not given by $SO(2,1)$. It should be mentioned here that $SU(1,1)$ in turn has a higher covering group which gives the complete set of representations, which has no simple realization. We shall confine ourselves to $SU(1,1)$ and later to only integral and half integral representations of $SU(1,1)$.

(i) We chose our carrier space to be the set of all 2×2 hermitian matrices with equal diagonal elements,(i.e)

$$\begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \tag{3.83}$$

Denoting the above matrix by X, and the group element by u, we define the group action as

$$X' = u^\dagger X u \tag{3.84}$$

Let us verify that the form of the matrix X is invariant under the above group action.

$$X'^\dagger = (u^\dagger X u)^\dagger = u^\dagger X^\dagger u = u^\dagger X u = X' \tag{3.85}$$

$$Tr(\sigma_3 X') = Tr(\sigma_3 u^\dagger X u) = Tr(X u \sigma_3 u^\dagger) = Tr(X' \sigma_3) = Tr(\sigma_3 X) \tag{3.86}$$

Thus X' is a member of the carrier space.

Now let us show that eqn[] satisfies the axioms of the group. The closure property is verified if

$$u = u_1 u_2 \tag{3.87}$$

(i.e) if u_1 and u_2 belongs to the group, then u also belongs to the group.

$$u\sigma u^\dagger = u_1 u_2 \sigma_3 (u_1 u_2)^\dagger \tag{3.88}$$

$$= u_1 u_2 \sigma_3 u_2^\dagger u_1^\dagger \tag{3.89}$$

$$= u_1 \sigma_3 u_1^\dagger \tag{3.90}$$

$$= \sigma_3 \tag{3.91}$$

Thus it satisfies the closure property. The identity element is δ^i_j and obeys

$$u.I = u \tag{3.92}$$

The inverse element of u is

$$u^{-1} = \sigma_3 u^\dagger \sigma_3 \tag{3.93}$$

(i.e)

$$u u^{-1} = u \sigma_3 u^\dagger \sigma_3 \tag{3.94}$$

$$= \sigma_3 \sigma_3 = I \tag{3.95}$$

(ii) To find the one parameter subgroups let us determine the mapping between $SO(2,1)$ and $SU(1,1)$. Considering the group actions in $SO(2,1)$ and $SU(1,1)$ we have

$$x' = ax \tag{3.96}$$

and

$$X' = u^{\dagger} X u \tag{3.97}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.98}$$

and

$$X = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \tag{3.99}$$

Using eqn[3.96] and eqn[3.97] the one parameter subgroups corresponding to the subgroups of $SO(2,1)$ given by eqn[3.72] turns out to be

$$\begin{aligned} u(\psi,\phi,\theta) = & \begin{pmatrix} \cosh\psi/2 & i\sinh\psi/2 \\ -i\sinh\psi/2 & \cosh\psi/2 \end{pmatrix} . \begin{pmatrix} \cosh\phi/2 & -\sinh\phi/2 \\ -\sinh\phi/2 & \cosh\phi/2 \end{pmatrix} \\ & . \begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix} \end{aligned} \tag{3.100}$$

Here unlike in $SU(2)$ we have used the subgroups of $SO(2,1)$ given in eqn.[3.72] without any change.

(iii) The infinitismal generators of the group are,

$$A_1 = \frac{\partial u}{\partial \psi} \mid_{\psi=\phi=\theta=0} = -\frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.101}$$

$$A_2 = \frac{\partial u}{\partial \phi} \mid_{\psi=\phi=\theta=0} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.102}$$

$$A_3 = \frac{\partial u}{\partial \theta} \mid_{\psi=\phi=\theta=0} = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.103}$$

Using sigma matrices and replacing A_i by K_i given by

$$K_i = -iA_i \tag{3.104}$$

we have

$$K_1 = \frac{i}{2}.\sigma_2 \tag{3.105}$$

$$K_2 = \frac{i}{2}.\sigma_1 \tag{3.106}$$

$$K_3 = \frac{1}{2}.\sigma_3 \tag{3.107}$$

To get the commutation relations in a neat form we interchange the labelling for K_1 and K_2 . (i.e)

$$K_1 = \frac{i}{2}.\sigma_1 \tag{3.108}$$

$$K_2 = \frac{i}{2}.\sigma_2 \tag{3.109}$$

$$K_3 = \frac{1}{2}.\sigma_3 \tag{3.110}$$

Now in terms of the infinitismal generators the generators of the group elements can be wrriten as

$$u(\psi, \phi, \theta) = \exp(iK_1\psi).\exp(iK_2\phi).\exp(iK_3\theta) \tag{3.111}$$

(iv) The commutation relations for the infinitismal generators are,

$$[K_1, K_2] = -iK_3 \tag{3.112}$$

$$[K_2, K_3] = iK_1 \tag{3.113}$$

$$[K_3, K_1] = iK_2 \tag{3.114}$$

Chapter 4

Realization of the SU(1,1) Group

4.1 Introduction

In this chapter we realize the groups as transformations on the function space.It should be noted that presently we do not realize it on a Hilbert space.By a function space we visualize a vector space whose elements are continuous functions. Hilbert space is a function space with a norm and a scaler product defined on it.In the following two sections we find a method for calculating the infinitismal generators of the operations defined in eqn[2.62] and eqn[2.63]. The result of these sections are then used in the following sections to find the generators of the groups $SO(3)$,$SU(2)$,$SO(2,1)$ and $SU(1,1)$ when realiazed as transformations on the function space.

4.2 Realization of the Group as Transformations on the Function Space

Consider the transformation,

$$g(x_i) = T(a_r, f(x_i,)) \tag{4.1}$$

where f and g are functions in the function space defined on the coordinate space variables x_i and a_i are the group parameters.We assume the function T to be analytic. The above transformation should satisfy,

$$g = T(a, f) \tag{4.2}$$

$$h = T(b, g) \tag{4.3}$$

$$h = T(c, f) \tag{4.4}$$

$$T(c, f) = T(b, T(a, f)) \tag{4.5}$$

$$= T(\phi(a, b), f) \tag{4.6}$$

Demanding the generators to take the exponential form (exactly as we proceeded in Chapter2), the transformations that will be dealt by us are

$$(i) \qquad g(x) = f(a^T.x) \tag{4.7}$$

$$(ii) \qquad g(x) = \mu(a, a^T.x).f(a^T.x) \tag{4.8}$$

where $x' = a^T.x$ is a linear transformation satisfying eqn.[3.6] of Chapter 2. In terms of the canonical parameters the matrix a is,

$$a = a(t_r) \tag{4.9}$$

The purpose of using the transpose of the matrix a, instead of the matrix itself will be soon shown explicitly below.It will be seen that only if we use the transpose of the matrix or the inverse of the matrix,the defined operation satisfies the closure property of the group.

From now on we shall use a better notation for the group action, namely

$$(i) \qquad g(x) \equiv T_a.f(x) = f(a^T.x) \tag{4.10}$$

$$(ii) \qquad g(x) \equiv T_a.f(x) = \mu(a, a^T.x).f(a^T.x) \tag{4.11}$$

We shall now explicity show that the above operations satisfy the group axioms. For the closure property we start with,

$$T_a.T_b.f(x) = T_a.f(b^T.x) \tag{4.12}$$

Denoting

$$f'(x) \equiv f(b^T.x) \tag{4.13}$$

we have

$$T_a.T_b.f(x) = T_a.f'(x) \tag{4.14}$$

$$= f'(a^T.x) \tag{4.15}$$

$$= f(b^T(a^T.x)) \tag{4.16}$$

$$= f((ab)^T.x) \tag{4.17}$$

$$= T_{ab}.f(x) \tag{4.18}$$

It can be easily varified that if the transpose (or the inverse) is not used , the operations don't satisfy the closure property.The closore property for the operation in eqn.[4.10] got by replacing a^T by a^{-1} , follows exactly as above.The above analysis for the operations in eqn.[4.11] will require us to define the binary operation in terms of the function $\mu(a, a^T.x)$ called 'multiplier' by Bargmann. We shall be dealing with this group action in the next section. The other group axioms for the operations in eqn.[4.10] is easily verified using the following relations.

$$T_{ab}.T_c.f(x) = T_a.T_{bc}.f(x) \tag{4.19}$$

$$T_a.T_e.f(x) = T_a.f(x) \tag{4.20}$$

$$T_a.T_{a^{-1}}.f(x) = T_e.f(x) \tag{4.21}$$

Once the operation (eqn.[4.10]) is verified to be a group action , our next venture should be to find its generators (and infinitismal generators). It would be a nice problem to determine the infinitismal generators for the general group action

$$T_a.f(x) = T(a, f(x)) \tag{4.22}$$

But we shall confine ourselves to the operation

$$T_a.f(x) = f(a^T.x) \tag{4.23}$$

for which the calculations are simpler.We know

$$\delta f = \frac{\partial f}{\partial x_i}.\delta x_i \tag{4.24}$$

and

$$x' = a^T(t_r).x \tag{4.25}$$

Comparing eqn.[4.25] with eqn.[3.6] and using eqn.[3.7] for the infitnitismal generators of the group ,when realized as transformations on the coordinate space, we have

$$x + \delta x = ex + \frac{\partial(a^T.x)}{\partial t_k}.\delta t^k \tag{4.26}$$

$$\delta x^i = \frac{\partial[(aT)^i_j.x^j]}{\partial t_k}.\delta t^k \tag{4.27}$$

$$= [(X_k)^i_j]^T.x^j.\delta t^k \tag{4.28}$$

Using this in eqn[4.24] we have

$$\delta f = \frac{\partial f}{\partial x_i}.[(X_k)^i_j]^T.x^j.\delta t^k \tag{4.29}$$

From the identities,

$$x^T.A^T.y = y^T.A.x \tag{4.30}$$

$$x \equiv x^i \qquad ; \qquad x^T \equiv x_i \tag{4.31}$$

in eqn.[4.29],we have

$$\delta f = \delta t_k.x_i.(X^k)^i_j.\frac{\partial f}{\partial x^j} \tag{4.32}$$

Thus the infinitismal generators denoted by A^k are ,

$$A^k = x_i.(X^k)^i_j.\frac{\partial}{\partial x^j} \tag{4.33}$$

where X^k is given by the expression eqn.[3.7].The generators for the group elements is got from eqn.[4.33] by the prescription given in eqn.[2.56].

$$T_a.f(x) = (\prod_k \exp(A_k.t_k)).f(x) \tag{4.34}$$

(i.e)

$$T_a \equiv \prod_k \exp(A_k.t_k) \tag{4.35}$$

From the infinitismal generators the commutation relations are calculated by blunt calculation.

4.3 Generalized Group Action

In this section we shall repeat the analysis done in the previous section for the operation

$$T_a.f(x) = \mu(a, a^T.x).f(a^T.x) \tag{4.36}$$

As mentioned before, the function $\mu(a, a^T.x)$ are called 'multipliers' .

We shall firstly verify that the above operation satisfies the group axioms.This as we shall see will require us to demand that the multipliers satisfy a set of multiplication laws. The closure property requires

$$T_a.T_b.f(x) = T_{ab}.f(x) \tag{4.37}$$

$$T_a.T_b.f(x) = T_a.[\mu(b, b^T.x).f(b^T.x)] \tag{4.38}$$

Denoting

$$f'(x) \equiv \mu(b, b^T.x).f(b^T.x) \tag{4.39}$$

$$T_a.T_b.f(x) = T_a.f'(x) \tag{4.40}$$

$$= \mu(a, a^T.x).f'(a^T.x) \tag{4.41}$$

$$= \mu(a, a^T.x).\mu(b, b^T.a^T.x).f(b^T.a^T.x) \tag{4.42}$$

$$= \mu(a, a^T.x).\mu(b, (ab)^T.x).f((ab)^T.x) \tag{4.43}$$

The R.H.S of eqn[4.37] is

$$T_{ab}.f(x) = \mu(ab, (ab)^T.x).f((ab)^T.x) \tag{4.44}$$

If the operation in eqn.[4.36] is to satisfy the closure property, the multipliers should satisfy

$$\mu(a, a^T.x).\mu(b, (ab)^T.x) = \mu(ab, (ab)^T.x) \tag{4.45}$$

Venturing into the associative law we have

$$T_{ab}.T_c.f(x) = T_a.T_{bc}.f(x) \tag{4.46}$$

this requires

$$\mu(ab, (ab)^T.x).\mu(c, (abc)^T.x) = \mu(a, a^T.x).\mu(bc, (abc)^T.x) \tag{4.47}$$

The other axioms are

$$T_a.T_e.f(x) = T_a.f(x) \tag{4.48}$$

and

$$T_a.T_{a^{-1}}.f(x) = T_e.f(x) \tag{4.49}$$

and these require,

$$\mu(e,x).\mu(a,a^T.x) = \mu(a,a^T.x) \tag{4.50}$$

and

$$\mu(a,a^T.x).\mu(a^{-1},x) = \mu(e,x) \tag{4.51}$$

We can now summarise the above results. The group action is defined as,

$$T_a.f(x) = \mu(a,a^T.x).f(a^T.x) \tag{4.52}$$

where the multipliers satisfies the following relations

$$\mu(a,a^T.x).\mu(b,(ab)^T.x) = \mu(ab,(ab)^T.x) \tag{4.53}$$

$$\mu(e,x).\mu(a,a^T.x) = \mu(a,a^T.x) \tag{4.54}$$

$$\mu(a,a^T.x).\mu(a^{-1},x) = \mu(e,x) \tag{4.55}$$

$$\mu(ab,(ab)^T.x).\mu(c,(abc)^T.x) = \mu(a,a^T.x).\mu(bc,(abc)^T.x) \tag{4.56}$$

Our next problem is to find the infinitismal generators for the group action. This calculation is slightly intricate and has been treated in a much more generalized way by Bargmann. We shall determine the same in a less concrete way. The approach uses the fact that the multipliers themselves form a group (eqn.[4.53]) and thus will have its infinitismal generators. We shall firstly determine the infinitismal grnerators of the multipliers and then use them to determine the infinitismal generators of the group action.

For the determination of the infinitismal generators of the multipliers , we start from

$$\mu(a,a^T.x) + \delta\mu = \mu(a.\delta a,(a.\delta a)^T.x) \tag{4.57}$$

Using eqn [4.53]

$$= \mu(a, a^T.x).\mu(\delta a, (a.\delta a)^T.x) \tag{4.58}$$

$$= \mu(a, a^T.x).[\mu(a, a^T.x) \mid_{a=e} + \frac{\partial \mu}{\partial a} \mid_{a=e} .\delta a] \tag{4.59}$$

$$= \mu(a, a^T.x).[\mu(e, x) + \frac{\partial \mu}{\partial a} \mid_{a=e} .\delta a] \tag{4.60}$$

$$= \mu(a, a^T.x) + \mu(a, a^T.x).\frac{\partial \mu}{\partial a} \mid_{a=e} .\delta a \tag{4.61}$$

therefore,

$$\delta\mu = \mu(a, a^T.x).\frac{\partial \mu}{\partial a} \mid_{a=e} \delta a \tag{4.62}$$

Noting that

$$a = a^i_j(t_r) \tag{4.63}$$

we have,

$$\delta\mu = \mu(a, a^T.x).\frac{\partial \mu}{\partial t^k} \mid_{t_k=e} .\delta t_k \tag{4.64}$$

Let us now look into the group action,

$$T_a.f(x) = \mu(a, a^T.x).f(a^T.x) \tag{4.65}$$

We can write this group action in the form

$$T_a.f(x) = \mu(a, a^T.x).T^0_a.f(x) \tag{4.66}$$

where T^0_a is the group action defined in eqn.[4.10] and which was studied in the last section. We can write

$$\delta[T_a.f(x)] = \mu(a, a^T.x).\delta[T^0_a.f(x)] + [\delta\mu(a, a^T.x)].T^0_a.f(x) \tag{4.67}$$

Using eqn.[4.33] and eqn.[4.64] in eqn[4.67] we have

$$\delta[T_a.f(x)] = \mu(a, a^T.x).[A^{0^k}\delta t_k].f(a^T.x) + \mu(a, a^T.x).\frac{\partial \mu}{\partial t^k} \mid_{t=0} .\delta t_k.f(a^T.x) \tag{4.68}$$

where A^{0^k} are infinitismal generators of T^0_a . Thus the infinitismal generators of the group action is

$$A^k = A^{0^k} + \frac{\partial \mu}{\partial t^k} \mid_{t=e} \tag{4.69}$$

$$= x_i.(X^k)^i_j.\frac{\partial}{\partial x_j} + \frac{\partial \mu}{\partial t^k} \mid t = e \tag{4.70}$$

where X^k is again given by eqn.[3.9]. Thus the infinitismal generators of the group action given in eqn[4.11] is the sum of the infinitismal generators of the group action given in eqn[4.10] and the infinitismal generators of the multiplier.

The generators of the group elements are got by exponentiating eqn[4.70].Thus the generator is

$$T_a = \prod_k \exp(A_k.t_k) \tag{4.71}$$

Finally the commutation relations are got by blunt calculation.

4.4 Realization of SO(3) and SU(2) as Transformations on the Function Space

In this section we shall use the results studied in sec.3.2 to find the generators and commutation relations of the $SO(3)$ and $SU(2)$ group when they are realized as transformations on the function space. The same study for the $SO(2,1)$ and $SU(1,1)$ group will be done in the following section.

The group action for the $SO(3)$ group is

$$T_a.f(x) = f(a^T.x) \tag{4.72}$$

where a is a 3×3 orthogonal matrix of determinant $+1$.It can be easily verified that the above group action satisfies the group axioms as was done in eqn.[4.37] to eqn[4.49].The infinitismal generators as per eqn.[4.33] are given by

$$J^k = x_i.(X^k)^i_j.\frac{\partial}{\partial x^j} \tag{4.73}$$

Using eqn.[3.17] for X^k, we have

$$J^1 = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} \tag{4.74}$$

$$J^2 = x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3} \tag{4.75}$$

$$J^3 = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} \tag{4.76}$$

The generator for the group elements are,

$$T_a = \exp(J^1\psi).\exp(J^2\phi).\exp(J^3\theta) \tag{4.77}$$

The commutation relations are,

$$[J^i, J^j] = -\epsilon_k^{ij}.J^k \tag{4.78}$$

Now let us consider the $SU(2)$ group.The group action is

$$T_u.f(z) = f(u^T.z) \tag{4.79}$$

where z is a vector in C^2 space and u is a 2×2 unitary matrix of determinant $+1$, of the form given in eqn[3.22].Again the group axioms can be easily verified. The infinitismal generators are got by using eqn.[3.47] in eqn[4.33].

$$J^1 = \frac{1}{2}(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1}) \tag{4.80}$$

$$J^2 = \frac{i}{2}(z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2}) \tag{4.81}$$

$$J^3 = \frac{1}{2}(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}) \tag{4.82}$$

At this stage we cannot talk about the hermitianity of the generators because we have not yet defined a scalar product. The generators of the group elements are

$$T_u = \exp(J^1\psi).\exp(J^2\phi).\exp(J^3\theta) \tag{4.83}$$

The commutation relations are,

$$[J^i, J^j] = i.\epsilon_k^{ij}.J^k \tag{4.84}$$

4.5 Realization of SO(2,1) and SU(1,1) as Transformations on the Function Space

The group action for the $SO(2,1)$ group is

$$T_a.f(x) = f(a^T.x) \tag{4.85}$$

where a is a 3×3 pseudo orthogonal matrix of determinant $+1$ satisfying eqn.[3.53]. It is easily verified that the above group action satisfies the group axioms. The infinitismal generators are determined by using eqn.[3.73] in eqn.[4.33].

$$K^1 = (x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_2}) \tag{4.86}$$

$$K^2 = -(x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3}) \tag{4.87}$$

$$K^3 = (x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}) \tag{4.88}$$

Comparing the generators of the $SO(2,1)$ and $SO(3)$ group, it is observed that both have a common generator. This is due to the fact that both $SO(3)$ and $SO(2,1)$ groups have $SO(2)$ as its subgroup. The generator of the group elements are

$$T_a = \exp(K^1\psi).\exp(K^2\phi).\exp(K^3\theta) \tag{4.89}$$

The commutation relations are

$$[K_1, K_2] = K_3 \tag{4.90}$$

$$[K_2, K_3] = -K_1 \tag{4.91}$$

$$[K_3, K_1] = -K_2 \tag{4.92}$$

For $SU(1,1)$ the group action is

$$T_u.f(z) = f(u^T.z) \tag{4.93}$$

where z is a vector in C^2 space and u is a 2×2 pseudo unitary matrix of determinant $+1$ satisfying eqn.[3.80]. The group axioms are easily verified. Using eqn.[3.105] in eqn.[4.33] we get the infinitismal generators as

$$K^1 = \frac{i}{2}(z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_1}) \tag{4.94}$$

$$K^2 = \frac{1}{2}(z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1}) \tag{4.95}$$

$$K^3 = \frac{1}{2}(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}) \tag{4.96}$$

Again comparing the generators of the $SU(1,1)$ and $SU(2)$ group we see that one of generators is common. The generator of the group elements are

$$T_u = \exp(K^1\psi).\exp(K^2\phi).\exp(K^3\theta) \tag{4.97}$$

The commutation relations are

$$[K_1, K_2] = -iK_3 \tag{4.98}$$

$$[K_2, K_3] = iK_1 \tag{4.99}$$

$$[K_3, K_1] = iK_2 \tag{4.100}$$

Chapter 5

UIR's of the SU(1,1) Group

5.1 Introduction

In this Chapter we work on the main part of our venture namely the representations of the group. Initially we define terms like group representation, unitary irreducible representation (UIR), etc. and some theorems which will be used in this Chapter.These are included for the sake of completeness.The details of these discussions can be found in any good book on group representation theory.

We shall classify all the UIR's of $SU(2)$ and $SU(1,1)$. Throughout this chapter we donot define the Hilbert space explicitly. We shall do the whole classification on an arbitrary Hilbert space with an arbitrary scalar product. The UIR's will be realized on explicitly defined Hilbert space in the next chapter.

5.2 Representation of a Group

In the earlier chapters we realized the groups as transformations on the coordinate space and function space. When a group is realized as transformations on a Hilbert space, the realization is called a Group representation.

Defnition of a group representation

It is a mapping $g \to T_g$ which associates with every element a of the group a linear operator T_a acting on a Hilbert space such that the linear operators form a group.

Definition of a Unitary Group Representation :

A unitary group representation is a group representation which satisfies

$$(f, g) = (T_a f, T_a g) \tag{5.1}$$

for any group element a and for any two members of the carrier space (Hilbert space) f and g. (f, g) denotes the scaler product in the Hilbert space.

Among the various possible forms of the scaler product ,we shall confine ourselves to the scaler product of the form,

$$(f, g) = \int \bar{f(x)}.g(x).w(x).dx \tag{5.2}$$

where $w(x)$ is called the weight factor. Since we demand the scaler product to be finite, the functions of the Hilbert space are called square integrable.

Theorem 4.1 :

Given a group representation of a compact group, there exists a scaler product in the Hilbert space, with respect to which the group representation is unitary.

It should be clearly noted that the above theorem which has a simple proof, is true for compact groups only. The above theorem has been proved for $SU(1,1)$ also and we shall use this theorem when we are dealing with $SU(1,1)$.

Theorem 4.2 :

Given a group representation of $SU(1,1)$,there exists a scaler product in the Hilbert space with respect to which the group representation is unitary.

Thus we need to deal with only the unitary group representations. All other group representations which are not unitary can always be made unitary by the above theorem using an appropriate similarity transformation. Since we shall be dealing with unitary representations we can work with the Dirac bra-ket notation. Thus we shall now denote the scalar product by

$$(f, g) = \ < f \mid g >\ \tag{5.3}$$

The infinitismal generators of a unitary representation are hermitian. This is concluded from the fact that any unitary operator can be written in terms of a hermitian operator as,

$$U = \exp(iH) \tag{5.4}$$

A hermitian operator satisfies,

$$(Hf, g) = (f, Hg) \tag{5.5}$$

5.3 Irreducible Representations

Let L_1 be a subspace of the carrier space (Hilbert space) L. For any element f of L_1 and for any element T_a of the group, if the transformed element

$$T_a.f \tag{5.6}$$

also belongs to the subspace L_1 ,then the subspace is called an invariant subspace.

Definition of a reducible Representation

If the carrier space of a group representation has atleast one invariant subspace other than the trivial one, then the group representation is called a reducible representation.

Definition of an Irreducible Representation

A group representation which is not reducible is called an irreducible group representation.

In general instead of the above definitions the well known criteria for irreducibility known as Schur's lemma is used.

Schur's Lemma

A group representation is irreducible, if and only if,any bounded linear operator A, which commutes with all the group elements is of the form

$$A = \alpha.I \tag{5.7}$$

where α is a complex number and I is the identity operator.

Since every group element can be written as a linear combination of the infinitismal generators, we can conclude that if there exists an operator A , which commutes with all infinitismal generators, then the representaion is irredulible and

$$A = \alpha.I \tag{5.8}$$

Our purpose is to find all the unitary irreducible represetations (UIR's) of a group. Using the above lemma our problem reduces to determination of independent operators(called invariant operators) which commutes with all the infinitismal generators of the group. We shall not be determining the invariant operators in a general manner here. Rather we shall assume the invariant operators of the group $SU(2)$ and $SU(1,1)$ to be given.

We need a couple of theorems to deal with the classification of the UIR's of $SU(2)$ and $SU(1,1)$ which will be our task in the next section.

Theorem 4.3

Every UIR of a compact group is finite dimensional.

Theorem 4.4

Every UIR of the $SU(1,1)$ group is infinite dimentional except for the trival one dimensional case.

5.4 Invariant Operators of SU(2) and SU(1,1)

In this section we shall discuss the invariant operators of $SU(2)$ and $SU(1,1)$ which will be used in the later sections. Both $SU(2)$ and $SU(1,1)$ sports one independent operator each. The invariant operators can be chosen in many forms and each form

can be related to the other . We shall discuss the form of the invariant operator which is most suitable for labelling the UIR's.

For $SU(2)$ the invariant operator is given by

$$Q = J_1^2 + J_2^2 + J_3^2 \tag{5.9}$$

It can be verified using eqn.[4.84] that Q commutates with J_1, J_2 and J_3.Using eqn.[4.80] the above invariant operator can be written in the form

$$Q = J(J+1) \tag{5.10}$$

where

$$J = \frac{1}{2}.(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) \tag{5.11}$$

It can again be verified that J also commutes with J_1, J_2 and J_3. Thus J is also an invariant operator. During classification of the UIR's of $SU(2)$ J will be seen to be a better lable than Q.

For $SU(1,1)$ the invariant operator is given by

$$Q = K_1^2 + K_2^2 + K_3^2 \tag{5.12}$$

Q commutes with K_1,K_2 and K_3 can be verified using eqn.[4.98]. Using eqn.[4.94] the above invariant operator can be written in the form

$$Q = K(1-K) \tag{5.13}$$

where

$$K = -\frac{1}{2}(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) \tag{5.14}$$

It can be verified that K also commutes with K_1, K_2 and K_3. Thus K is also an invariant operator.During classification of the UIR's of the $SU(1,1)$ group K will be found to be a better label than Q.

5.5 Classification of UIR's of SU(2)

If Q is an invariant of the group, then Schur's lemma tells us that

$$Q = q.I \tag{5.15}$$

where q is a complex number. In general q is restricted to take certain values only. Thus if we determine the possible value of q, then q can be used as a lable for the UIR's. Thus our aim would be to find the values that can be attained by q.

As was mentioned in the introduction we shall not be specifying our Hilbert space explicity.Let J_x, J_y and J_z be the hermitian infinitismal generators of a unitary representation of the $SU(2)$ group. Again since the Hilbert space is not explicitly specified,we shall simply work with J_x, J_y and J_z and shall not worry about its explicit form. Using the fact that the eigen vectors of J_z form a complete orthogonal set of basis vectors, we can write

$$J_z \mid \lambda_0 + n >= (\lambda_0 + n) \mid \lambda_0 + n > \tag{5.16}$$

where $\mid \lambda_0 + n >$ denotes the eigen vectors in the Direc notation. The possible values of λ_0 and n are

$$-1/2 < \lambda_0 \leq 1/2 \tag{5.17}$$

$$n = 0, \pm 1, \pm 2, ... \tag{5.18}$$

Presently we have used this typical notation to stress that from the start we cannot be sure that $\mid 0 >$ is an eigenvector.Later ofcourse for $SU(2)$ λ_0 turns out to be 0 or $1/2$. In $SU(1,1)$ which will be dealt in the next section it will be seen that λ_0 can take all values in the specified domain.

From the commutation relations determined in eqn.[3.50] it can be easily checked that

$$Q = J_x^2 + J_y^2 + J_z^2 \tag{5.19}$$

is an invariant operator. Thus from Schur's lemma,

$$Q = q.I \tag{5.20}$$

where q is a complex number.(i.e)

$$Q \mid \lambda_0 + n >= q \mid \lambda_0 + n > \tag{5.21}$$

Thus q is the eigen value of the invariant operator Q. Since J_x, J_y and J_z are hermitian operators their eigen values a, b and c respectively will be real. Thus

$$q = a^2 + b^2 + c^2 \geq 0 \tag{5.22}$$

(i.e)

$$q \geq 0 \tag{5.23}$$

Thus q is surely a positive real quantity.

Before proceeding further we shall change over to another set of infinitismal generators. We define them as,

$$J_z = J_z \tag{5.24}$$

$$J_+ = J_x + iJ_y \tag{5.25}$$

$$J_- = J_x - iJ_y \tag{5.26}$$

The commutation relation satisfied by these are

$$[J_z, J_+] = J_+ \tag{5.27}$$

$$[J_z, J_-] = -J_- \tag{5.28}$$

$$[J_+, J_-] = 2.J_z \tag{5.29}$$

It should be noted that J_+ and J_- are not hermitian, they satisfy

$$(J_+)^\dagger = J_- \tag{5.30}$$

We shall now see that J_+ and J_- act as ladder operators. Starting from eqn.[5.27] we have

$$J_z J_+ - J_+ J_z = J_+ \tag{5.31}$$

$$J_z J_+ \mid \lambda_0 + n > -J_+ J_z \mid \lambda_0 + n >= J_+ \mid \lambda_0 + n > \tag{5.32}$$

Using eqn.[5.16]

$$J_z[J_+ \mid \lambda_0 + n >] = (\lambda_0 + n + 1)[J_+ \mid \lambda_0 + n >] \tag{5.33}$$

again using eqn.[5.16] we conclude

$$J_+ \mid \lambda_0 + n >=\mid \lambda_0 + n + 1 > \tag{5.34}$$

except for a proportionality constant. Similarly starting from the eqn.[5.28] we get

$$J_- \mid \lambda_0 + n >=\mid \lambda_0 + n - 1 > \tag{5.35}$$

again the proportionality constant is avoided.

Letting the proportionality constants to be c_n and d_n, we shall soon find a relation between them. Let

$$J_+ \mid \lambda_0 + n >= c_n \mid \lambda_0 + n + 1 > \tag{5.36}$$

and

$$J_- \mid \lambda_0 + n >= d_n \mid \lambda + n - 1 > \tag{5.37}$$

These can be wrtten as

$$< \lambda_0 + n + 1 \mid J_+ \mid \lambda_0 + n >= c_n \tag{5.38}$$

$$< \lambda_0 + n - 1 \mid J_- \mid \lambda_0 + n >= d_n \tag{5.39}$$

Using eqn.[5.16] we have

$$c_n^* = (J_+)^\dagger =< \lambda_0 + n \mid (J_+)^\dagger \mid \lambda_0 + n + 1 > \tag{5.40}$$

$$=< \lambda_0 + n \mid J_- \mid \lambda_0 + n + 1 > \tag{5.41}$$

$$= d_{n+1} \tag{5.42}$$

Thus,

$$d_n = c_{n-1}^* \tag{5.43}$$

Therefore eqn.[5.36] and eqn.[5.37] can be written as

$$J_+ \mid \lambda_0 + n >= c_n \mid \lambda_0 + n + 1 > \tag{5.44}$$

$$J_- \mid \lambda_0 + n >= c_{n-1}^* \mid \lambda_0 + n - 1 > \tag{5.45}$$

Our idea in $SU(1,1)$ will be to use the fact that $\mid c_n \mid^2$ is a positive real quantity for any n. Presently we start from the commutation relations in eqn.[5.29].

$$[J_+, J_-] = 2.J_z \tag{5.46}$$

$$J_+J_- \mid \lambda_0 + n > -J_-J_+ \mid \lambda_0 + n >= 2.J_z \mid \lambda_0 + n > \tag{5.47}$$

using eqn.[5.44] and eqn.[5.45]

$$c_{n-1}.c_{n-1}^* - c_n.c_n^* = 2.(\lambda_0 + n) \tag{5.48}$$

$$| c_{n-1} |^2 - | c_n |^2 = 2.(\lambda_0 + n) \tag{5.49}$$

This is a difference equation and its solution is given by

$$| c_n |^2 = A - (\lambda_0 + n)(\lambda_0 + n + 1) \tag{5.50}$$

where A is an arbitary constant. Using the relation

$$J_+ J_- + J_- J_+ = 2.Q - 2.J_z^2 \tag{5.51}$$

which can be easily verified using eqn.[5.9]. Thus

$$2.Q \mid \lambda_0 + n >= J_+ J_- \mid \lambda_0 + n > + J_- J_+ \mid \lambda_0 + n > -2.J_z^2 \mid \lambda_0 + n > \tag{5.52}$$

$$2q = | c_{n-1} |^2 + | c_n |^2 + 2(\lambda_0 + n)^2 \tag{5.53}$$

$$= 2A \tag{5.54}$$

(i.e)

$$q = A \tag{5.55}$$

Using eqn.[5.55] and eqn[5.50] we have

$$| c_n |^2 = q - (\lambda_0 + n)(\lambda_0 + n + 1) \tag{5.56}$$

At this stage Theorem 4.3 simplifies the calculation for $SU(2)$ compared to $SU(1,1)$. Finite dimentionality of the UIR's of $SU(2)$ implies that the set of eigen values has a maximum and a minimum. Denoting it by $(\lambda_0 + n_>)$ and $(\lambda_0 + n_<)$, we require

$$J_+ \mid \lambda_0 + n_> >= 0 \tag{5.57}$$

and

$$J_- \mid \lambda_0 + n_< >= 0 \tag{5.58}$$

which is equivalent to demanding

$$| c_{n_>} |^2 = | c_{n_<-1} |^2 = 0 \tag{5.59}$$

using eqn.[5.56] we have

$$q - (\lambda_0 + n_>)(\lambda_0 + n_> + 1) = 0 \tag{5.60}$$

$$q - (\lambda_0 + n_< - 1)(\lambda_0 + n_<) = 0 \tag{5.61}$$

Solutions of these equations are

$$q = (\lambda_0 + n_>)(\lambda_0 + n_> + 1) \tag{5.62}$$

$$(\lambda_0 + n_>) = -(\lambda_0 + n_<) \tag{5.63}$$

Since the difference between the maximum eigen value and the minimum eigen value is equal to the number of eigen vectors of J_z we have,

$$(\lambda_0 + n_>) - (\lambda_0 + n_<) \tag{5.64}$$

to be a nonnegative integer.Using eqn.[5.63] we have $2(\lambda_0 + n_>)$ to be a nonnegative integer. Denoting $(\lambda_0 + n_>)$ by j and $(\lambda_0 + n)$ by m, we have

$$q = j(j + 1) \tag{5.65}$$

$$j = 0, \frac{1}{2}, 1, ... \tag{5.66}$$

Thus we can lable each of the UIR's of $SU(2)$ by j, where j is a nonnegative integer or half integer and in each UIR the eigen states of J_z are

$$-j \leq m \leq j \tag{5.67}$$

5.6 Classification of UIR's of SU(1,1)

Our approach will be exactly the same as was done for $SU(2)$. We shall start with the invariant operator Q for $SU(1,1)$ and find the values that can be attained by its eigen values. These values are used as lables for the UIR's of $SU(1,1)$. For $SU(1,1)$ it turns out that these values by themselves are not enough for labelling the UIR's. We shall soon see that another quantity λ_0 is necessary for classifying and labelling the UIR's completely. Again we shall not be specifying our Hilbert space explicity.

Starting from the fact that the eigen vectors of J_z forms a complete orthogonal set of basis vectors, we write

$$J_z \mid \lambda_0 + n >= (\lambda_0 + n) \mid \lambda_0 + n > \tag{5.68}$$

where $\mid \lambda_0 + n >$ are the eigen vectors of J_z. λ_0 and n can attain

$$-1/2 < \lambda_0 \leq 1/2 \tag{5.69}$$

$$n = 0, \pm 1, \pm 2, ... \tag{5.70}$$

The invariant operator for the $SU(1,1)$ group is

$$Q = K_x^2 + K_y^2 + K_z^2 \tag{5.71}$$

Thus from Schur's lemma,

$$Q = q.I \tag{5.72}$$

(i.e)

$$Q \mid \lambda_0 + n >= q \mid \lambda_0 + n > \tag{5.73}$$

where q is a complex number. Since J_x, J_y and J_z are hermitian operators, their eigen values are real and thus we can conclude

$$q \;\; is \;\; real \tag{5.74}$$

Defining the infinitismal generators to be

$$K_z = K_z \tag{5.75}$$

$$K_+ = K_x + iK_y \tag{5.76}$$

$$K_- = K_x - iK_y \tag{5.77}$$

The commutation relations satisfied by these are,

$$[K_z, K_+] = K_+ \tag{5.78}$$

$$[K_z, K_-] = -K_- \tag{5.79}$$

$$[K_+, K_-] = -2.K_z \tag{5.80}$$

Also we have

$$(K_+)^\dagger = K_- \tag{5.81}$$

It is easily observed that eqns.[5.78] , [5.79] and [5.80] are exactly the same as for $SU(2)$ (eqns.[5.27] , [5.28] and [5.29]). Thus the analysis for $SU(2)$ from eqn.[5.31] to [5.45] turns out to be exactly the same for $SU(1,1)$ again. Thus we can conclude that K_+ and K_- act as ladder operators and satisfy

$$K_+ \mid \lambda_0 + n >= c_n \mid \lambda_0 + n + 1 > \tag{5.82}$$

$$K_- \mid \lambda_0 + n >= c^*_{n-1} \mid \lambda_0 + n - 1 > \tag{5.83}$$

Our next aim will be to get the analogous relation of eqn.[5.56] (for $SU(2)$)in $Su(1,1)$. Starting with eqn.[5.80]

$$[K_+, K_-] = -2.K_z \tag{5.84}$$

$$K_+K_- \mid \lambda_0 + n > -K_-K_+ \mid \lambda_0 + n >= -2.K_z \mid \lambda_0 + n > \tag{5.85}$$

using eqn.[5.82] and [5.83]

$$\mid c_{n-1} \mid^2 = \mid c_n \mid^2 = -2.(\lambda_0 + n) \tag{5.86}$$

The solution to this difference equation is

$$\mid c_n \mid^2 = A + (\lambda_0 + n)(\lambda_0 + n + 1) \tag{5.87}$$

where A is an arbitary constant. Using the relation

$$K_+K_- + K_-K_+ = 2Q + 2K_z^2 \tag{5.88}$$

which is easily verified using eqn.[5.71].Thus

$$2Q \mid \lambda_0 + n >= K_+K_- \mid \lambda_0 + n > +K_-K_+ \mid \lambda_0 + n > -2K_z^2 \mid \lambda_0 + n > \tag{5.89}$$

$$2q = \mid c_{n-1} \mid^2 + \mid c_n \mid^2 - 2(\lambda_0 + n)^2 \tag{5.90}$$

$$= 2A \tag{5.91}$$

(i.e)

$$q = A \tag{5.92}$$

Thus eqn.[5.87] is

$$| c_n |^2 = q + (\lambda_0 + n)(\lambda_0 + n + 1) \tag{5.93}$$

Noting that c_n is a complex number, we have the condition

$$| c_n |^2 \geq 0 \qquad ,\forall \quad n \tag{5.94}$$

(i.e)

$$q + (\lambda_0 + n)(\lambda_0 + n + 1) \geq 0 \qquad ,\forall \quad n \tag{5.95}$$

Classification of the UIR's :

At this stage our problem comes down to the determination of all sets of values of λ_0,q and n which satisfy eqn.[5.95]. It is easily seen that for every possible value of λ_0 in the domain $-1/2 < \lambda_0 \leq 1/2$,we can find an appropriate set of values of q and n which satisfies eqn.[5.95]. At this stage when we have not specified the Hilbert space explicity this seems possible. But when we venture into the specification of the Hilbert space explicity ,it becomes conceptually very tough to deal with cases other than $\lambda_0 = 0$ and $\lambda_0 = 1/2$. It turns out that the UIR's other than $\lambda_0 = 0$ and $\lambda_0 = 1/2$ can be realized explicity on a Hilbert space only if we consider the covering group of $SU(1,1)$ namely the group $\bar{SU}(1,1)$. This group it seems cannot be visualized as a linear transformation on the coordinate space. Thus due to the conceptual toughness of the problem, and seeing into the fact that the persons of the highest calibre in the subject like V-Bargmann and I.M.Gelfand have confined themselves to the cases $\lambda_0 = 0$ and $\lambda_0 = 1/2$, we shall also confine ourselves to the same.

Restricting ourselves to $\lambda_0 = 0, 1/2$, we have

$$| c_n |^2 = q + n(n+1) \quad , \quad for \quad \lambda_0 = 0 \tag{5.96}$$

and

$$| c_n |^2 = q + (n + \frac{1}{2})(n + \frac{3}{2}) \quad , \quad for \quad \lambda_0 = \frac{1}{2} \tag{5.97}$$

These can be written as a single relation by denoting

$$\lambda_0 + n \equiv m \tag{5.98}$$

and letting m take all integral and half integral values.(i.e)

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.99}$$

$$forall \qquad m = 0, \pm\frac{1}{2}, \pm 1, ... \tag{5.100}$$

This if required can be broken as , for $\lambda_0 = 0$,

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.101}$$

$$forall \qquad m = 0, \pm 1, \pm 2, ... \tag{5.102}$$

and for $\lambda_0 = 1/2$,

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.103}$$

$$for \ \ all \qquad m = \pm\frac{1}{2}, \pm\frac{3}{2}, ... \tag{5.104}$$

Equipped with eqn.[5.99] we shall now classify the UIR's of $SU(1,1)$.

Continuous Class :

This class derives its name from the fact that in this class q takes a continuous set of values in a domain.It will be seen that $\lambda_0 = 1/2$ corresponds to a single class while $\lambda_0 = 0$ case can be broken into two classes. Finally we combine two of the above classes to get two continuous classes namely the Principal series and the Supplementary series.

$\underline{\lambda_0 = 0}$

From eqn.[5.101]

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.105}$$

$$forall \qquad m = 0, \pm 1, \pm 2, ... \tag{5.106}$$

The above condition can be true for any $m = 0, \pm 1, \pm 2, ...$ only if q satisfies

$$q \geq 0 \tag{5.107}$$

Thus this class corresponds to

$$\lambda_0 = 0; \qquad q \geq 0; \qquad m = 0, \pm 1, \pm 2, ... \tag{5.108}$$

Let us now analyse the condition $q \geq 0$ in detail.Using the invariant operator defined in eqn.[5.13] our condition becomes

$$k(1-k) \geq 0 \tag{5.109}$$

where k is the eigen value of k and satisfies an eigen value equation similar to eqn.[5.73].Though q was shown to be a non-negative real constant, k can take on complex values also. We shall now divide the continuous class for $\lambda_0 = 0$ into two classes based on whether k is real or not.For non-real k

$$k(1-k) \geq 0 \tag{5.110}$$

(i.e)

$$\bar{k} = 1 - k \tag{5.111}$$

Therefore

$$Re(k) = \frac{1}{2} \tag{5.112}$$

Thus we can write

$$k = \frac{1}{2} + i\ s \qquad , s \in R \tag{5.113}$$

We also have

$$q = (\frac{1}{2} + is)(1 - \frac{1}{2} - is) \qquad , s \in R \tag{5.114}$$

$$= \frac{1}{4} + s^2 \qquad , s \in R \tag{5.115}$$

$$\geq \frac{1}{4} \tag{5.116}$$

Thus confining ourselves to non real k is equivalent to restricting ourselves to $q \geq \frac{1}{4}$ .Thus we lable our above class by the following restrictions

$$\begin{array}{lll} \lambda_0 = 0 & ; & q \geq \frac{1}{4} \quad ; \\ k = \frac{1}{2} + is \quad , s \in R & ; & \\ m = 0, \pm 1, \pm 2, ... & & \end{array} \tag{5.117}$$

We now consider the case when k is real.

$$k(1-k) \geq 0 \tag{5.118}$$

(i.e)

$$(k+\frac{1}{2}-\frac{1}{2})(\frac{1}{2}+\frac{1}{2}-k) \geq 0 \tag{5.119}$$

Therefore

$$(k-\frac{1}{2}) \geq \frac{1}{4} \tag{5.120}$$

Thus we can write

$$k = \frac{1}{2}+r \qquad , \quad -\frac{1}{2} \leq \frac{1}{2} \tag{5.121}$$

We also have

$$q = (\frac{1}{2}+r)(1-\frac{1}{2}-r) \qquad , \quad -\frac{1}{2} \leq r \leq \frac{1}{2} \tag{5.122}$$

$$= \frac{1}{4}-r^2 \qquad , \quad -\frac{1}{2} \leq r \leq \frac{1}{2} \tag{5.123}$$

(i.e)

$$0 \leq q \leq \frac{1}{4} \tag{5.124}$$

Thus real k confines ourselves to $0 \leq q \leq \frac{1}{4}$ .Thus our second class corresponding to $\lambda_0 = 0$ is

$$\begin{array}{lll} \lambda_0 = 0 & ; & 0 \leq q \leq \frac{1}{4} \quad ; \\ k = \frac{1}{2}+r \quad , \quad -\frac{1}{2} \leq r \leq \frac{1}{2} & ; & \\ m = 0, \pm 1, \pm 2, ... & & \end{array} \tag{5.125}$$

We shall next deal with the case $\lambda_0 = \frac{1}{2}$

$\underline{\lambda_0 = \frac{1}{2}}$

From eqn.[5.103]

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.126}$$

$$forall \qquad m = \pm\frac{1}{2}, \pm\frac{3}{2}, ... \tag{5.127}$$

The above condition can be true for any $m = \pm\frac{1}{2}, \pm\frac{1}{2}, ...$ only if q satisfies

$$q \geq \frac{1}{4} \tag{5.128}$$

The above condition can also be reached from eqn.[5.97] .

$$q = (n + \frac{1}{2})(n + \frac{3}{2}) \geq 0 \quad , \forall \quad n = 0, \pm 1, \pm 2, ... \tag{5.129}$$

(i.e)

$$(q - \frac{1}{4}) + (n + 1)2 \geq 0 \quad , \forall \quad n = 0, \pm 1, \pm 2, ... \tag{5.130}$$

Therefore

$$q \geq \frac{1}{4} \tag{5.131}$$

Thus the class corresponds to

$$\lambda - 0 = \frac{1}{2} \quad ; q \geq \frac{1}{4} \quad ; m = \pm\frac{1}{2}, \pm\frac{3}{2}, ... \tag{5.132}$$

In eqn.[5.116] we had concluded that $q \geq \frac{1}{4}$ corresponds to non real k. Thus our class corresponding to $\lambda_0 = 1/2$ is

$$\begin{array}{lll} \lambda_0 = \frac{1}{2} & ; & q \geq \frac{1}{4} \quad ; \\ k = \frac{1}{2} + is \quad , & s \in R & ; \\ m = \pm\frac{1}{2}, \pm\frac{3}{2}, ... & & \end{array} \tag{5.133}$$

Thus we have described three classes in the continuous class,two corresponding to $\lambda_0 = 0$ and one corresponding to $\lambda_0 = 1/2$. Looking into the specifications of the classes listed in eqn.[5.117], [5.125] , and [5.133], we observe that [5.117] and [5.133] have all the specifications same except for the values that m can attain. Remembering that m corresponds to the eigen vectors of the UIR, it is clear that [5.117] and [5.133] can be shunted into a single class. This new class after the shunting is called the Principal series. Similarly the class corresponding to eqn.[5.125] is called the Supplementary series. Thus we have the continuous classes as

(i) Principal series :

$$\begin{array}{lll} \lambda_0 = 0, 1/2 & ; & q \geq \frac{1}{4} \quad ; \\ k = \frac{1}{2} + is \quad , & s \in R & ; \\ m = 0, \pm\frac{1}{2}, \pm 1, ... & & \end{array} \tag{5.134}$$

(ii) Supplementary series :

$$\begin{array}{lll} \lambda_0 = 1/2 \quad ; & & 0 \leq q \leq \frac{1}{4} \quad ; \\ k = \frac{1}{2} + r \quad , & -\frac{1}{2} \leq r \leq \frac{1}{2} \quad ; & \\ m = 0, \pm 1, \pm 2, ... & & \end{array} \tag{5.135}$$

An intresting observation in the above classification is that the UIR corresponding to $\lambda_0 = 1/2; k = 1/2$ is common to both the principal and the supplementary series.

Discrete class:

This class derives its name from the fact that q is restricted to take discrete values in its domine. In the continuous class to get infinite dimentional UIR's we had let m extend from negative infinity to postive infinity. But we can have infinite dimentional UIR's if m extends to infinity only in one direction and terminates on the other direction. The termination of n will be seen to give conditions which restricts q to discrete values.

Clearly we can have two classes of representations. One in which m extends to negative infinity and has a finite maximum values and another in which m extends to the positive infinity and has a finite minimum value. The former is called the negative discrete series and the later the positive discrete series.

Let us start with the negative discrete series.Starting with eqn.[5.99], we have

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.136}$$

$$forall \qquad , m = 0, \pm\frac{1}{2}, \pm 1, ... \tag{5.137}$$

Using the condition for negative discrete series that m has a maximum value we have

$$\mid c_m \mid^2 = q + m(m+1) \geq 0 \tag{5.138}$$

$$forall \qquad , m = 0, \pm\frac{1}{2}, \pm 1, ... \tag{5.139}$$

$$and \qquad m < m_> \tag{5.140}$$

The condition in eqn.[5.140] gives

$$\mid c_{m_>} \mid^2 = q + m_>(m_> + 1) = 0 \tag{5.141}$$

(i.e)

$$q = -m_>(m_> + 1) \tag{5.142}$$

Using this in eqn.[5.136] we have

$$\mid c_m \mid^2 = -m_>(m_> + 1) + m(m + 1) \geq 0 \tag{5.143}$$

$$forall \qquad , m = 0, \pm\frac{1}{2}, \pm 1, ... \tag{5.144}$$

$$and \qquad m \leq m_> \tag{5.145}$$

At the first instance seeing at eqn.[5.143] it may seem that $m_>$ can take on any integral or half integral values.But we shall see that positive integral or half integral values for $m_>$ donot satisfy the above conditions and $m_> = 0$ corresponds to the trival one dimentional representation. Thus only negative integral or half integral values for $m_>$ are allowed.

Consider the case $m_> > 0$ .For $m = m_> - 1$ eqn.[5.143] takes the form

$$\mid c_{m_>-1} \mid^2 = -m_>(m_> + 1) + (m_> - 1)m_> \tag{5.146}$$

$$= -2.m_> \quad < \quad 0 \tag{5.147}$$

Thus it fails to satisfy eqn.[5.143] and thus these values are discraded. Next for the case $m_> = 0$, eqn.[5.143] is

$$\mid c_m \mid^2 = m(m + 1) \geq 0 \tag{5.148}$$

$$forall \qquad , m = 0, \pm\frac{1}{2}, \pm 1, ... \tag{5.149}$$

$$and \quad m < 0 \tag{5.150}$$

Remembering that

$$K_- \mid m >= c^*_{m-1} \mid m - 1 > \tag{5.151}$$

we have

$$K_- \mid 0 >= c^*_{-1} \mid -1 > \tag{5.152}$$

$$= 0 \tag{5.153}$$

Thus all eigen vectors except $\mid 0 >$ turn out to be zero in this case. Thus for $m_> = 0$ we have a single eigen vector and so we have the trival one dimentional UIR for $m_> = 0$. Finally for the case $m_> < 0$ we show that the condition [5.143] is satisfied. Let s be a positive integer, then we have

$$\mid c_{m_>-s} \mid^2 = -m_>(m_> + 1) + (m_> - s)(m_> - s + 1) \tag{5.154}$$

$$= -2m_>s - s + s^2 \geq 0 \tag{5.155}$$

We can thus describe the discrete class by

$$m_> = 0, -\frac{1}{2}, -1, ... \tag{5.156}$$

$$m = m_>, m_> - 1, m_> - 2, ... \tag{5.157}$$

Remembering that $q = k(1 - k)$ where k is the invariant operator, we denote

$$-m_> = k \tag{5.158}$$

Thus the discrete class is

$$\begin{array}{l} \lambda_0 = 0, 1/2 \\ k = \frac{1}{2}, 1, \frac{3}{2}, ... \\ q = \frac{1}{4}, 0, -\frac{3}{4}, ... \\ m = -k, -k - 1, -k - 2, ... \end{array} \tag{5.159}$$

where we have excluded the trival one dimentional UIR.

An exactly similar analysis for the positive discrete series gives

$$\begin{array}{l} \lambda_0 = 0, 1/2 \\ k = \frac{1}{2}, 1, \frac{3}{2}, ... \\ q = \frac{1}{4}, 0, -\frac{3}{4}, ... \\ m = k, k + 1, k + 2, ... \end{array} \tag{5.160}$$

Thus we can summarize the UIR's of the discrete class as

(i) Trival one dimentional UIR :

$$\begin{array}{lll} \lambda_0 = 0, 1/2 & ; & k = 0 \quad ; \\ q = 0 \quad ; & & m = 0 \end{array} \tag{5.161}$$

(ii) Negative Discrete series :

$$\begin{array}{l} \lambda_0 = 0, 1/2 \\ k = \frac{1}{2}, 1, \frac{3}{2}, ... \\ q = \frac{1}{4}, 0, -\frac{3}{4}, ... \\ m = -k, -k-1, -k-2, ... \end{array} \tag{5.162}$$

(iii) Positive Discrete series :

$$\begin{array}{l} \lambda_0 = 0, 1/2 \\ k = \frac{1}{2}, 1, \frac{3}{2}, ... \\ q = \frac{1}{4}, 0, -\frac{3}{4}, ... \\ m = k, k+1, k+2, ... \end{array} \tag{5.163}$$

Listng all the UIR's of the SU(1,1) group,

(I) Continuous Class :

(i) Principal series :

$$\begin{array}{lll} \lambda_0 = 0, 1/2 \quad ; & & q \geq \frac{1}{4} \quad ; \\ k = \frac{1}{2} + is \quad , \quad s \in R \quad ; & & \\ m = 0, \pm\frac{1}{2}, \pm 1, ... & & \end{array} \tag{5.164}$$

(ii) Supplementary series :

$$\begin{array}{lll} \lambda_0 = 1/2 \quad ; & & 0 \leq q \leq \frac{1}{4} \quad ; \\ k = \frac{1}{2} + r \quad , \quad -\frac{1}{2} \leq r \leq \frac{1}{2} \quad ; & & \\ m = 0, \pm 1, \pm 2, ... & & \end{array} \tag{5.165}$$

(II) Discrete Class :

(i) Trival one dimentional UIR :

$$\begin{array}{lll} \lambda_0 = 0, 1/2 \quad ; & k = 0 \quad ; \\ q = 0 \quad ; & m = 0 \end{array} \tag{5.166}$$

(ii) Negative discrete series :

$$\begin{array}{l} \lambda_0 = 0, 1/2 \\ k = \frac{1}{2}, 1, \frac{3}{2}, ... \\ q = \frac{1}{4}, 0, -\frac{3}{4}, ... \\ m = -k, -k-1, -k-2, ... \end{array} \tag{5.167}$$

Positive Discrete series :

$$\begin{array}{l} \lambda_0 = 0, 1/2 \\ k = \frac{1}{2}, 1, \frac{3}{2}, ... \\ q = \frac{1}{4}, 0, -\frac{3}{4}, ... \\ m = k, k+1, k+2, ... \end{array} \tag{5.168}$$

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